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ON ALTERNATIVE REPRESENTATIONS OF TIME VARYING VISCOELASTIC MATERIALS†

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Abstract—Conditions for equivalence between linear differential and a class of integral operators are given and the results are devoted to applications to alternative representations of linear, time varying viscoelastic materials. Several examples are presented, including a discussion on sufficient conditions under which time varying linear differential operators can be used to analytically represent asymptotically stable materials, and an example on the inversion of integral equations appearing in the treatment of thermorheologically simple materials.

1. INTRODUCTION. REMARKS ON THE REPRESENTATION OF GENERAL VISCOELASTIC MATERIALS

THE literature on representation of general viscoelastic materials by means of linear, time-varying operators is scarce and was mainly devoted in the past to materials such as concrete [1], whose mechanical properties are not time-invariant due to aging processes induced by chemical hardening, moisture diffusion, etc. At present, those operators are progressively attracting the attention of other areas in view of its application to linearized theories of thermo-elasticity and thermo-viscoelasticity.

In general, most of the currently available research lies in the area of integral equations, mainly because integral equations are a natural way of formulating the physical problem. In some important applications, the step response (or its derivative) can experimentally be determined furnishing in this way the kernel of the integral operator, thus reducing considerably the analytical procedures of the identification problem.

Apart from the classical work of Arutiunian [1], the literature on differential constitutive equations seems to be scarce. Recent work is available [2] wherein a second order differential equation is investigated and [3, 4], where differential equations are derived from consideration of time-varying, spring-dashpots, models.

Interest in a complementary alternative approach arises from many different points of view. Besides systematic aspects, indeed the most important reason is the necessity of an approach exhibiting the advantages of an easy formulation with the possibility of easy numerical solutions. And while the solution of several hundreds, first order ordinary differential equations subject to initial values is a routine matter for a digital computer, the solution of systems of Volterra integral equations by means of the usual quadrature procedures may easily exceed the rapid access storage capacity of contemporary computers. On the other hand, a combination of the two approaches, integral and differential, makes tractable and feasible the solution of several important identification problems associated with linear and nonlinear viscoelasticity. Problems of this type will be presented separately.

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2. MATHEMATICAL BACKGROUND

It is well known that the solution of linear ordinary differential equations subject to initial conditions can be reduced to the solution of a class of linear Volterra integral equations. There are several methods to accomplish this purpose, notably the so called Fubini method (Tricomi [5]) [6], which is found to be of great interest in the investigation of asymptotic solutions of differential equations. The theory of linear systems also furnishes very expressive examples of this possibility. For example, every solution of the system

$$\frac{dz}{dt} = Az + B(t)z, \tag{2.1}$$

satisfies a linear Volterra integral equation

$$z = y + \int_0^t Y(t-\tau)B(\tau)z(\tau)\,d\tau, \tag{2.2}$$

where y and Y are the vector and matrix solution respectively of the reduced, non perturbed, ($B \equiv 0$) equation and where $y(0) = z(0)$ and $Y(0) = I$, the identity matrix (Bellman [7]).

Conversely, Evans [8] reduced the solutions of a class of Volterra integral equations to the solution of a certain differential equation subject to appropriate initial conditions by considering the class of Volterra equations whose kernels satisfy certain linear differential equations. This idea seems to have been first considered by Evans although little attention in the author's opinion has been paid to this interesting possibility.

In the same line, Teodone [9] and Volterra [10] considered the class of integral equations whose inversion requires only a finite number of integrals and derivatives. This turns out to be an interesting point of discussion nowadays when accurate digital computers are available and when the idea of approximate solutions under suitable metrics replaces the necessity of solutions obtained by means of elementary operators. In this sense, Bellman *et al.* [11] introduced the idea of differential approximation for the solution of equations of the type

$$x = y + \int_0^t f(t-\tau)x(\tau)\,d\tau, \tag{2.3}$$

obtained in the course of constructing some mathematical models of physiological processes connected with cancer chemotherapy. Here the kernel f is assumed to satisfy an ordinary, generally nonlinear, differential equation. The solution x is then obtained as the solution of an approximate linear differential equation, the approximation defined by a suitable norm, subject to appropriate initial conditions which in turn depend on the given function y.

In this paper, equivalence of the representations given by

$$y = Cx + \int_0^t f(t,\tau)x(\tau)\,d\tau \tag{2.4}$$

and

$$a_0 y + a_1 y^{(1)} + \ldots + y^{(N)} = b_0 x + b_1 x^{(1)} + \ldots + b_N x^{(N)} \tag{2.5}$$

is investigated and the results are devoted to applications of alternative, differential and integral, representations of linear visco-elastic materials. Several examples involving specialized representations are presented in Section 6. In Section 7, the method is applied to finding sufficient conditions for the representation of several types of asymptotically stable materials. Finally, the algorithm is applied to the numerical solution of integral equations typical in the theory of thermorheologically simple materials. Further extensions of the method, in connection with the identification problem, will be discussed elsewhere.

3. FORMULATION OF THE EQUATIONS

Let $C(t, \tau)$, $\tau \leq t$, be the response (creep function) of a general linear (time varying) material submitted to a unit step stress $\sigma(t) = H(t-\tau)$ starting at time τ. Then the response $\varepsilon(t)$ (strain) to a given stress history $\sigma(t)$ is given by

$$\varepsilon(t) = \int_{-\infty}^{t} C(t, \tau)\,\mathrm{d}\sigma(\tau). \tag{3.1}$$

It will be assumed here that $C(t, \tau)$ possesses a discontinuity of the first kind (finite jump) at $\tau = t^{+}$. It will also be assumed that the material is in a quiescent state prior to an instant $t = 0$ and that any stress history $\sigma(t)$ and its derivatives up to the Nth order are continuous except possibly at a finite number of points where finite jumps are allowed.

Integrating equation (3.1) by parts and taking into account the restrictions imposed above, the function $\varepsilon(t)$ may be written

$$\varepsilon(t) = C(t, t^{+})\sigma(t) + \int_{0}^{t} f(t, \tau)\sigma(\tau)\,\mathrm{d}\tau, \tag{3.2}$$

where

$$f(t, \tau) = -\frac{\partial}{\partial \tau} C(t, \tau) \tag{3.3}$$

is the "impulse creep function" of the material. The term $C(t, t^{+})\sigma(t)$ accounts for the delta function singularity introduced by the derivative of $C(t, \tau)$ at $\tau = t^{+}$.

An alternative, differential, representation of a linear, time varying viscoelastic material can be given by the following N-order differential equation

$$a_0(t)\varepsilon + a_1(t)\frac{\mathrm{d}\varepsilon}{\mathrm{d}t} + \ldots + \frac{\mathrm{d}^N\varepsilon}{\mathrm{d}t^N} = b_0(t)\sigma + b_1(t)\frac{\mathrm{d}\sigma}{\mathrm{d}t} + \ldots + b_N(t)\frac{\mathrm{d}^N\sigma}{\mathrm{d}t^N}. \tag{3.4}$$

The principal aim of this paper is to investigate conditions on a_i, b_i and $C(t, \tau)$ under which the representations given by the equations (3.2) and (3.4) are equivalent and to derive an algorithm to obtain one representation when the other is given. In this line, let us first consider an example which without loss in generality, exhibits the essence of the method proposed here.

Consider the integral representation given by equation (3.2). In most applications the kernel $f(t, \tau)$ of the integral equation can be conveniently approximated by the sum of products of functions t and τ. Consider for instance, the class of kernels which may be written in the form

$$f(t, \tau) = \varphi_1(\tau)F_1(t) + \varphi_2(\tau)F_2(t). \tag{3.5}$$

This representation suggests the possibility of considering $f(t, \tau)$ as the solution of second order partial differential equation of the type

$$a_0(t)f(t,\tau)+a_1(t)\frac{\partial f(t,\tau)}{\partial t}+\frac{\partial^2 f(t,\tau)}{\partial t^2} = 0, \tag{3.6}$$

subject to suitable initial conditions given by

$$\begin{aligned} [f(t,\tau)]_{\tau=t} &= \alpha_0(t), \\ \left[\frac{\partial f(t,\tau)}{\partial t}\right]_{\tau=t} &= \alpha_1(t), \end{aligned} \tag{3.7}$$

where $\alpha_0(t)$ and $\alpha_1(t)$ are given functions of t.

Clearly, by substitution of $f(t, \tau)$ given by equation (3.5) in the differential equation (3.6), it is recognized that $F_1(t)$ and $F_2(t)$ are two solutions of the following second order ordinary differential equation

$$a_0(t)F(t)+a_1(t)\frac{\mathrm{d}F}{\mathrm{d}t}+\frac{\mathrm{d}^2F}{\mathrm{d}t^2} = 0, \tag{3.8}$$

and that functions φ_1 and φ_2 must satisfy the following system of linear equations

$$\begin{aligned} \varphi_1(t)F_1(t)+\varphi_2(t)F_2(t) &= \alpha_0(t), \\ \varphi_1(t)\frac{\mathrm{d}F_1(t)}{\mathrm{d}t}+\varphi_2(t)\frac{\mathrm{d}F_2(t)}{\mathrm{d}t} &= \alpha_1(t), \end{aligned} \tag{3.9}$$

obtained by direct substitution of $f(t, \tau)$ given by equation (3.5) in equation (3.7). If, for given functions α_0 and α_1, the functions φ_1 and φ_2 are required to be unambiguously determined, then F_1 and F_2 must be any two *linearly independent solutions* of the ordinary differential equation (3.8).

Now, differentiating equation (3.2) twice, forming the sum

$$\sum_{i=0}^{2} a_i(t)\frac{\mathrm{d}^i\varepsilon}{\mathrm{d}t^i}, \qquad a_2 = 1,$$

and taking into account equation (3.6), the following differential equation is obtained

$$a_0(t)\varepsilon(t)+a_1(t)\frac{\mathrm{d}\varepsilon(t)}{\mathrm{d}t}+\frac{\mathrm{d}^2\varepsilon(t)}{\mathrm{d}t^2} = b_0(t)\sigma(t)+b_1(t)\frac{\mathrm{d}\sigma(t)}{\mathrm{d}t}+b_2(t)\frac{\mathrm{d}^2\sigma(t)}{\mathrm{d}t^2} \tag{3.10}$$

where

$$\begin{aligned} b_0(t) &= a_0C+a_1\frac{\mathrm{d}C}{\mathrm{d}t}+\frac{\mathrm{d}^2C}{\mathrm{d}t^2}+a_1f(t,t)+\frac{\mathrm{d}f(t,t)}{\mathrm{d}t}+\left(\frac{\partial f(t,\tau)}{\partial t}\right)_{\tau=t}, \\ b_1(t) &= a_1C+2\frac{\mathrm{d}C}{\mathrm{d}t}+f(t,t), \\ b_2(t) &= C, \end{aligned} \tag{3.11}$$

and where C stands for $C(t, t^+)$.

This example shows the equivalence between the representations given by equations (3.2) and (3.10) when certain conditions have been fulfilled. It also shows how to construct the kernel of the integral equation when the differential equation is given, and conversely. In fact, suppose that C and the kernel $f(t, \tau)$ given by equation (3.5) are known functions. Then a_0 and a_1 appearing in equations (3.8) and (3.10) can be calculated from the system

$$\begin{aligned} a_0(t)F_1(t)+a_1(t)\frac{dF_1}{dt} &= -\frac{d^2F_1}{dt^2}, \\ a_0(t)F_2(t)+a_1(t)\frac{dF_2}{dt} &= -\frac{d^2F_2}{dt^2}, \end{aligned} \tag{3.12}$$

and b_0, b_1 and b_2 appearing in equation (3.10) can be directly evaluated by means of equation (3.11).

Conversely, given the differential equation (3.10), $\varepsilon(t)$ admits the representation given by equation (3.2), where

$$C(t, t^+) = b_2(t), \tag{3.13}$$

and $f(t, \tau)$ is given by equation (3.5) where F_1 and F_2 are any two linear independent solutions of equations (3.8) and the functions φ_1 and φ_2 can be evaluated from equation (3.9). Functions α_0 and α_1 appearing in equation (3.9) can be recursively evaluated using equation (3.11), yielding

$$\begin{aligned} \alpha_0(t) &= [f(t,\tau)]_{\tau=t} = b_1-\left(a_1b_2+2\frac{db_2}{dt}\right), \\ \alpha_1(t) &= \left[\frac{\partial f(t,\tau)}{\partial t}\right]_{\tau=t} = b_0-\left(a_0b_2+a_1\frac{db_2}{dt}+\frac{d^2b_2}{dt^2}\right) \\ &\quad -a_1b_1+a_1\left(a_1b_2+2\frac{db_2}{dt}\right)-\frac{db_1}{dt}+\frac{d}{dt}\left(a_1b_2+2\frac{db_2}{dt}\right). \end{aligned} \tag{3.14}$$

4. GENERALIZATION. KERNELS OF ORDER N

The formal procedure used above can be generalized to order N under reasonable conditions on functions a_i, b_i and C.

Let $a_i(t)$, $i = 0, 1, \ldots, N$, $a_N \equiv 1$, be a set of N real functions, $C(t, \tau) \equiv 0$ if $t < \tau$, and such that it satisfies the following partial differential equation

$$\sum_{i=0}^{N} a_i(t)\frac{\partial^i}{\partial t^i}\frac{\partial C(t,\tau)}{\partial \tau} = 0, \qquad a_N \equiv 1. \tag{4.1}$$

Under such conditions function $C(t, \tau)$ is defined to be an *Nth order creep function* and the associated function

$$f(t,\tau) = -\frac{\partial C(t,\tau)}{\partial \tau}, \tag{4.2}$$

is defined to be a *kernel of order N*.

In what follows, the following "natural" initial conditions associated with the differential equation (4.1) are considered

$$C(t,t^+) = b_N(t)$$
$$\left[\frac{\partial^{i-1} f(t,\tau)}{\partial t^{i-1}}\right]_{\tau=t} = \alpha_{i-1}, \qquad i = 1, 2, \ldots, N, \tag{4.3}$$

where functions α_{i-1} are obtained from the following recurrence relations

$$\alpha_{i-1}(t) = b_{N-i} - \sum_{j=0}^{i} a_{N-j}\binom{N-j}{i-j}\frac{\mathrm{d}^{i-j}C(t,t^+)}{\mathrm{d}t^{i-j}}$$
$$-\sum_{j=1}^{i} a_{N-j}\sum_{\nu=1}^{i-j-1}\binom{N-i+\nu}{\nu}\frac{\mathrm{d}^{\nu}\alpha_{i-j-\nu-1}}{\mathrm{d}t^{\nu}}, \qquad i = 1, 2, \ldots, N, \tag{4.4}$$

where $\sum_j^i \equiv 0$ if $i < j$ and where b_i, $i = 0, 1, \ldots, N$ is a set of $N+1$ arbitrary real functions.

Now, given the differential equation

$$\sum_{i=0}^{N} a_i(t)\frac{\mathrm{d}^i\varepsilon}{\mathrm{d}t^i} = \sum_{i=0}^{N} b_i(t)\frac{\mathrm{d}^i\sigma}{\mathrm{d}t^i}, \tag{4.5}$$

and the Nth order creep function $C(t,\tau)$ which satisfies the initial conditions given by the equations (4.3) and (4.4), then

$$\varepsilon(t) = C(t,t^+)\sigma(t) + \int_0^t f(t,\tau)\sigma(\tau)\,\mathrm{d}\tau, \tag{4.6}$$

and conversely.

It will be first proved that if the Nth order creep function $C(t,\tau)$ satisfies the initial conditions given by equations (4.3) and (4.4), then equation (4.6) implies equation (4.5). In fact, using equation (4.3), equation (4.4) yields the following equation for b_i in terms of functions a_i,

$$b_i(t) = \sum_{j=1}^{N}\binom{j}{j-i}a_j\frac{\mathrm{d}^{j-i}C(t,t^+)}{\mathrm{d}t^{j-i}} + \sum_{j=i+1}^{N} a_j\sum_{\nu=0}^{j-i-1}\binom{i+\nu}{\nu}\frac{\mathrm{d}^{\nu}}{\mathrm{d}t^{\nu}}\left[\frac{\partial^{j-i-\nu-1}f(t,\tau)}{\partial t^{j-i-\nu-1}}\right]_{\tau=t}, \tag{4.7}$$

$i = 0, 1, \ldots, N$.

Differentiating $\varepsilon(t)$ given by equation (4.6) up to the ith order, multiplying by $a_i(t)$, forming the sum $\sum_{i=0}^N a_i(\mathrm{d}^i\varepsilon/\mathrm{d}t^i)$ and taking into account equations (4.1), (4.2) and (4.7), equation (4.5) is finally obtained.

The proof is completed by observing that the representation given by equation (4.6) is unique as it follows from the unicity of the solution of the partial differential equation (4.1) subject to the initial conditions given by equations (4.3) and (4.4).

5. CONSTRUCTION OF Nth ORDER CREEP FUNCTIONS

Let $F_i(t)$, $i = 1, 2, \ldots, N$, be a set of N linearly independent solutions of the following ordinary differential equation

$$\sum_{i=0}^{N} a_i(t)\frac{\mathrm{d}^i F(t)}{\mathrm{d}t^i} = 0, \qquad a_N \equiv 1. \tag{5.1}$$

Let $\varphi_i(t)$, $i = 0, 1, \ldots, N$, be a set of $N+1$ arbitrary functions. Then

$$C(t, \tau) = \varphi_0(t) + \sum_{i=1}^{N} [\Psi_i(t) - \Psi_i(\tau)]F_i(t), \tag{5.2}$$

where

$$\Psi_i(t) = \Psi_i(\tau) + \int_\tau^t \varphi_i(\xi)\,\mathrm{d}\xi, \qquad i = 1, 2, \ldots, N,$$

is an Nth order creep function as may be proved by substitution in equation (4.1). Recalling equation (4.2), the associated influence creep function results

$$f(t, \tau) = \sum_{i=1}^{N} \varphi_i(\tau)F_i(t). \tag{5.3}$$

If the representations given by the equations (4.5) and (4.6) are to be equivalent, then $C(t, \tau)$ and $f(t, \tau)$ given by equations (5.2) and (5.3) respectively, must satisfy the initial conditions given by equations (4.3) and (4.4) furnishing in this manner a set of $N+1$ equations from where it is possible to evaluate functions $\varphi_i(t)$. From equation (5.2) and the first equation (4.3), it follows

$$\varphi_0(t) = b_N(t). \tag{5.4}$$

Now, introducing the following N-dimensional vectors

$$\varphi = (\varphi_i), \qquad i = 1, 2, \ldots, N, \tag{5.5}$$

and

$$\alpha = (\alpha_i), \qquad i = 0, 1, \ldots, N-1, \tag{5.6}$$

and the matrix

$$W = \left(\frac{\mathrm{d}^{i-1}F_j}{\mathrm{d}t^{i-1}}\right), \qquad i, j = 1, 2, \ldots, N, \tag{5.7}$$

substitution of equation (5.3) in the second equation (4.3) yields

$$W\varphi = \alpha, \tag{5.8}$$

an equation which allows for the evaluation of the remaining N functions $\varphi_1, \varphi_2, \ldots, \varphi_N$ unambiguously, since det. W is the Wronskian of a set of N linearly independent functions.

6. APPLICATIONS. SPECIAL REPRESENTATIONS

(a) *Functions* a_i, $i = 0, 1, \ldots, N$, $a_N \equiv 1$, *are constants*

When the coefficients a_i are constants, a useful alternative representation of an N-order creep function $C(t, \tau)$ is given by

$$C(t,\tau) = \Phi_0(t) + \sum_{i=1}^{N} \int_{\tau}^{t} \Phi_i(\xi) F_i(t-\xi)\,\mathrm{d}\xi, \tag{6.1}$$

where Φ_i, $i = 0, 1, \ldots, N$, is a set of $N+1$ arbitrary functions, as may be proved by substitution in equation (4.1). The associated influence creep function is given by

$$f(t,\tau) = \sum_{i=1}^{N} \Phi_i(\tau) F_i(t-\tau). \tag{6.2}$$

Now, let ρ_i, $i = 1, 2, \ldots, N$ be the roots of the characteristic polynomial

$$a_0 + a_1\rho + \ldots + \rho^N = 0, \tag{6.3}$$

then F_i in equations (6.1) and (6.2) are given by

$$F_i(t) = \mathrm{e}^{\rho_i t}. \tag{6.4}$$

With regard to functions Φ_i appearing in equations (6.1) and (6.2), from the first equation (4.3) we obtain

$$\Phi_0(t) = b_N(t), \tag{6.5}$$

and from the second equation (4.3), and taking into account equations (6.2) and (6.4), we obtain the following system of equations

$$\begin{aligned} &\Phi_1 + \Phi_2 + \ldots \Phi_N = \alpha_0(t), \\ &\rho_1\Phi_1 + \rho_2\Phi_2 + \ldots + \rho_N\Phi_N = \alpha_1(t) \\ &\vdots \\ &\rho_1^{N-1}\Phi_1 + \rho_2^{N-1}\Phi_2 + \ldots + \rho_N^{N-1}\Phi_N = \alpha_{N-1}(t), \end{aligned} \tag{6.6}$$

from where the remaining functions Φ_i, $i = 1, 2, \ldots, N$, can be evaluated. In system (6.6), functions α_i are given by equations (4.4), if due account is taken that the a_i's are constants.

The expression for $C(t, \tau)$ given by equation (6.1) can be slightly modified so as to obtain a representation which does not explicitly contain integrals. Let $\lambda_i(t)$, $i = 1, 2, \ldots, N$ be N functions given by

$$\lambda_i(t) = \int^{t} \Phi_i(\xi) \mathrm{e}^{\rho_i(t-\xi)}\,\mathrm{d}\xi, \tag{6.7}$$

then

$$\frac{\mathrm{d}\lambda_i}{\mathrm{d}t} - \rho_i\lambda_i = \Phi_i, \qquad i = 1, 2, \ldots, N. \tag{6.8}$$

Substitution of Φ_i given by equation (6.8) in equation (6.1) and integration by parts yields the result

$$C(t,\tau) = \Phi_0(t) + \sum_{i=1}^{N} \lambda_i(t) - \sum_{i=1}^{N} \lambda_i(\tau) \mathrm{e}^{\rho_i(t-\tau)}. \tag{6.9}$$

When $a_0 \equiv 0$ and the polynomial

$$a_1 + a_2\rho + \ldots + \rho^{N-1} = 0$$

is Hurwitz, and if for convenience we set $\rho_1 = 0$, $\delta_i = -\rho_i > 0$, $i = 2, 3, \ldots, N$, then $C(t, \tau)$ given by equation (6.9) reduces to

$$C(t, \tau) = \Phi_0(\tau) + \sum_{i=2}^{N} \lambda_i(t)(1 - e^{-\delta_i(t-\tau)}), \tag{6.10}$$

provided

$$\Phi_0(t) + \sum_{i=1}^{N} \lambda_i(t) = 0. \tag{6.11}$$

Under appropriate restrictions on functions Φ_0 and $\lambda_i(t)$, $C(t, \tau)$ given by equation (6.10) was introduced by Arutiunian [1] to represent the creep of concrete.

(b) *Coefficients a_i and b_i are constants*

The case when not only the a_i's but also the b_i's are constants, leads to consideration of time-invariable materials, as might be expected. The integral equation reduces in this case to one of the convolution type. If the characteristic polynomial is Hurwitz, the creep function and the influence creep function are given by

$$C(t-\tau) = \Phi_0 - \sum_{i=1}^{N} \frac{\Phi_i}{\rho_i}(1 - e^{\rho_i(t-\tau)}), \tag{6.12}$$

$$f(t-\tau) = \sum_{i=1}^{N} \Phi_i e^{\rho_i(t-\tau)}, \tag{6.13}$$

respectively, where Φ_i, $i = 0, 1, \ldots, N$ are constants which must satisfy equations (6.5) and (6.6), taking into consideration that a_i, b_i and consequently α_i, $i = 0, 1, \ldots, N-1$ appearing in equation (6.6) are constants. In particular, the following known relations [12]—derived from equation (4.7)—

$$\begin{aligned} b_0 &= a_0 c_0 + a_1 f_0 + a_2 f_0^{(1)} + \ldots + f_0^{(n-1)}, \\ b_1 &= \qquad a_1 c_0 + a_2 f_0 \quad + \ldots + f_0^{(n-2)} \\ &\vdots \\ b_{N-1} &= \qquad\qquad a_{N-1} c_0 + f_0, \\ b_N &= \qquad\qquad\qquad c_0, \end{aligned} \tag{6.14}$$

hold between $a_i, b_i, i = 0, 1, \ldots, N, c_0 = C(0)$ and the successive derivatives of the kernel $f(t)$ at $t = 0$,

$$f_0^{(j)} = \left[\frac{d^j f(t)}{dt^j}\right]_{t=0}, \qquad j = 0, 1, \ldots, N-1.$$

(c) $a_i = b_i \equiv 0$, $i = 0, 1, \ldots, N-1$, $a_N \equiv 1$, $b_N \neq 0$

In this case, the differential equation (4.5) reduces to

$$\frac{d^N \varepsilon(t)}{dt^N} = b_N(t) \frac{d^N \sigma(t)}{dt^N}. \tag{6.15}$$

The associated kernel is immediately found to be

$$f(t,\tau) = \varphi_1(\tau)+\varphi_2(\tau)(t-\tau)+\ldots+\varphi_N(\tau)(t-\tau)^{N-1}. \tag{6.16}$$

Substitution of $f(t,\tau)$ given by equation (6.16) in equation (4.3) yields the following conditions for functions φ_i, $i = 1,2,\ldots,N$,

$$\varphi_i(t) = \alpha_{i-1}(t), \qquad i = 1,2,\ldots,N, \tag{6.17}$$

where α_i are given by equation (4.4), reducing in this case to be

$$\alpha_{i-1}(t) = -\binom{N}{i}\frac{\mathrm{d}^i b_N(t)}{\mathrm{d}t^i}, \qquad i = 1,2,\ldots,N. \tag{6.18}$$

Substitution of equations (6.17) and (6.18) in equation (6.16) yields

$$f(t,\tau) = -\left[Nb_N^{(1)}+\binom{N}{2}(t-\tau)b_N^{(2)}+\ldots+(t-\tau)^{N-1}b_N^{(N)}\right]. \tag{6.19}$$

7. ASYMPTOTICALLY STABLE MATERIALS

Let

$$\varepsilon(t) = C(t,t^+)\sigma(t)+\int_\tau^t f(t,\xi)\sigma(\xi)\,\mathrm{d}\xi \tag{7.1}$$

and

$$a_0\varepsilon+a_1\varepsilon^{(1)}+\ldots\varepsilon^{(N)} = b_0\sigma+b_1\sigma^{(1)}+\ldots+b_N\sigma^{(N)}, \tag{7.2}$$

be two equivalent representations of a given viscoelastic material, i.e. $C(t,\tau)$ is an N-order creep function which satisfies the initial conditions given by equations (4.3) and (4.4). In some applications, conditions are required such that for all bounded inputs $\sigma(t)$, $\varepsilon(t)$ is bounded as $t\to\infty$, for all $\tau\le t$. From inspection of equation (7.1), a necessary and sufficient condition for boundedness is

$$C(t,\tau) = \int_\tau^t f(t,\xi)\,\mathrm{d}\xi < \infty, \tag{7.3}$$

as $t\to\infty$.

The converse problem, i.e. for all $\varepsilon(t)$ bounded, under which conditions $\sigma(t)$ is bounded, is one of great interest and difficulties, related to a class of Tauberian theorems. Several results, including deterministic and stochastic versions of the problem, are available [13]. Here we do not pursue the same path. Using the formalism presented in previous sections and several standard results in the theory of ordinary differential equations, we find conditions for stability of differential equations of the type (7.2), which appear to be of interest in the construction of differential constitutive equations of asymptotically stable time varying materials.

Given the differential equation (7.2), equation (7.3) implies that the problem is then equivalent to finding conditions under which the solutions of the partial differential equation (4.1), subject to the initial conditions (4.3), are bounded. This problem is in turn intimately related to the investigation of the asymptotic behavior of the solutions of the differential equation (5.1). A detailed examination of this and related matters is beyond the scope of this paper. Many results are available in the literature on the stability and asymptotic behavior of ordinary differential equations. See for instance [5, 7, 14]. In the spirit of Fubini method, a memoire by Ghizzetti [6] can be mentioned. In this memoire, a remarkable criteria for stability is given, which may be considered a perturbation theorem with respect to the classical Hurwitz criteria when the coefficients are constants.

Here we shall only consider two examples which appear to be of interest in applications. Let

$$F = (F_i), \qquad i = 1, 2, \ldots, N, \tag{7.4}$$

be an N-dimensional vector whose components are linearly independent solutions of the differential equation (5.1). Now, recalling equations (5.3), (5.4) and (5.8), equation (7.3) can be written

$$C(t, \tau) = b_N(t) + F^T(t) \int_\tau^t W^{-1}(\xi)\alpha(\xi)\,\mathrm{d}\xi \tag{7.5}$$

where superscript T denotes transpose. Suppose now that the limits

$$\begin{aligned} \lim a_i(t) &= a_{i\infty}, \qquad i = 0, 1, \ldots, N, \\ \lim b_i(t) &= b_{i\infty}, \qquad i = 0, 1, \ldots, N, \end{aligned} \tag{7.6}$$

exist and that

$$a_{0\infty} + \rho a_{1\infty} + \ldots + \rho^N = 0, \tag{7.7}$$

is a Hurwitz polynomial. Then the functions $F_i(t)$ and its derivatives to the order $N-1$ will tend exponentially to zero as $t \to \infty$, and in consequence the limit of $C(t, \tau)$ as $t \to \infty$ given by equation (7.5) will reduce to

$$\lim_{t \to \infty} C(t, \tau) = b_{N\infty}, \tag{7.8}$$

independent of τ.

If equation (7.7) is no longer required to be a Hurwitz polynomial, then the limit of $C(t, \tau)$ as $t \to \infty$ will exhibit in general a dependence with the initial time τ. Consider for instance the case where $a_0 \equiv 0$ and $a_i, i = 1, 2, \ldots, N-1, a_N = i$, are constants. Let

$$a_1 + a_2\rho + \ldots + \rho^{N-1} = 0, \tag{7.9}$$

be a Hurwitz polynomial. Then a linearly independent set of solutions of equation (5.1) is found to be

$$F(t) = (1, \mathrm{e}^{\rho_2 t}, \mathrm{e}^{\rho_3 t}, \ldots, \mathrm{e}^{\rho_N t}), \tag{7.10}$$

where $\rho_2, \rho_3, \ldots, \rho_N$ are the real negative roots of equation (7.9). Recalling equation (6.2) and taking into account equation (7.10), $C(t, \tau)$ is given by

$$C(t, \tau) = b_N(t) + \int_\tau^t \varphi_1(\xi)\,\mathrm{d}\xi + \sum_{i=2}^N \int_\tau^t \varphi_i(\xi)\mathrm{e}^{\rho_i(t-\xi)}\,\mathrm{d}\xi, \tag{7.11}$$

where the functions φ_i are given by the solution of system (6.6). Assuming that functions $b_i(t)$ possess finite asymptotic limits $b_{i\infty}$, functions φ_i will also tend asymptotically to a finite limit $\varphi_{i\infty}$ as $t \to \infty$. Then

$$\lim_{t \to \infty} C(t, \tau) = b_{N\infty} + \int_\tau^\infty \varphi_1(\xi)\,\mathrm{d}\xi - \sum_{i=2}^N \frac{1}{\rho_i}\varphi_{i\infty}, \tag{7.12}$$

which express the fact that the asymptotic value of $C(t, \tau)$ as $t \to \infty$ relies upon the existence of the integral

$$\int_{\tau}^{\infty} \varphi_1(\xi)\,\mathrm{d}\xi < \infty. \tag{7.13}$$

If the coefficients a_i are not constants but

$$a_{1\infty}+a_{2\infty}\rho+\ldots\rho^{N-1} = 0, \tag{7.14}$$

is a Hurwitz polynomial and

$$\lim_{t\to\infty} a_0(t) = 0, \tag{7.15}$$

then the existence of a limit for function $C(t, \tau)$ will still depend on the existence of the integral

$$\int_{\tau}^{\infty} \varphi_1(\xi)\,\mathrm{d}\xi < \infty \tag{7.16}$$

where φ_1 is the first element of vector φ given by equation (5.8).

8. SOLUTION OF INTEGRAL EQUATIONS—EXAMPLE

In the theory of thermorheologically simple materials, integral equations of the type

$$u(t)+\int_0^t \frac{\partial}{\partial\tau} f[\alpha(t)-\alpha(\tau)]u(\tau)\,\mathrm{d}\tau = v(t), \tag{8.1}$$

are frequently encountered. An approximate differential equation can be found by assuming that $g(\xi)$ approximating $f(\xi)$ under certain norm, satisfies a linear differential equation

$$\sum_{i=0}^{N} a_i g^{(i)} = 0, \tag{8.2}$$

and applying some of the results presented above. We omit here a discussion on approximation aspects, which will be presented separately in connection with the identification problem, and restrict ourselves to the following numerical example.

Let u be the unknown function in the integral equation

$$u(t)+\int_0^t \alpha'(\tau)\mathrm{e}^{-[\alpha(t)-\alpha(\tau)]}u(\tau)\,\mathrm{d}\tau = v(t), \tag{8.3}$$

where

$$v(t) = (1+\alpha)\mathrm{e}^{-\alpha}, \tag{8.4}$$

and α is a function given by the differential equation

$$\alpha' = 2-\mathrm{e}^{-t^2}, \qquad \alpha(0) = 0. \tag{8.5}$$

A solution to (8.3) and (8.4) is found to be

$$u(t) = \mathrm{e}^{-\alpha(t)}, \tag{8.6}$$

as it can be proved by direct substitution.

Since the kernel in equation (8.3) is already given as a product of functions of t and τ, by differentiation, equation (8.3) can be reduced to the following first order differential equation

$$u' = -2\alpha' u + v' + \alpha' v, \qquad u(0) = 1. \tag{8.7}$$

Equations (8.4), (8.5) and (8.7) constitute a straightforward initial-value problem which was numerically solved using a Runge–Kutta scheme on a CDC 6400, with step size 0·01. Results agreed in 9 or 10 significant figures with the exact values, in the interval $0 \leq t \leq 1$. Execution took less than a second.

9. DISCUSSION

The foregoing results were derived for the simplest, linear uniaxial case, to emphasize the structure of the underlying algorithms. It is noted, however, that with no essential modifications, they still hold in higher finite dimensional spaces. It is also noted that this approach is not restricted only to linear operators, but embraces also Volterra integral equations of the type

$$x(t) = y(t) + \int_0^t f(t, \tau) G[x(\tau)] \, d\tau$$

where G is a nonlinear function of the n-dimensional vector x. This and related matters will be discussed elsewhere.

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Абстракт—Даются условия эквивалентности между линейными дифференциальными операторами и некоторым классом интегральных операторов. Результаты применяются для варианта представлений линейных, зависящих от времени, вязкоупругих материалов. Представляются несколько примеров, заключая обсуждение достаточных условий, при которых можно пользоватьоя линейными дифференциальными, зависящими от времени, операторами для аналитического представления аоимптотически стабильных материалов. Дается, также, пример инверсии интегральных уравнений, которые встречаются в обсуждении термореологических, простых материалов.